LIBOR and Swap Market Models
for Pricing Interest Rate Derivatives
Monte Carlo Simulations

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To my father Godefroy Bidima Ndengte,
May his Soul rest in Peace.
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Abstract

After reviewing essential tools that constitute the basis of Financial Calculus, we derive the stochastic differential equations modelling LIBOR interest rates and Swap market instruments under several equivalent martingale measures and in particular under the Terminal measure. We give an account of pricing interest rate derivatives such as Caps and Swaptions within these market models. We illustrate finally the use of Monte Carlo Methods in Finance by calculating numerically the caplet payoffs for a given cap contract within the LIBOR Market Models.

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Introduction

Time is money. If you borrow an amount of money today, you will certainly not pay back the same amount tomorrow. You will surely pay it with some interest. The interest rate quoted over the period of the loan is more often not fixed, but depends on time and randomly on the state of the world. This gives a class of interest rates known as \textit{stochastic interest rates}. Many models had been invented in financial markets to describe the dynamics of such interest rates in continuous time. The ones introduced recently by Miltersen, Sandmann and Sondermann (1997), and Brace, Gatarek and Musiela (1997) appear useful and more practical to the real-world market. They are called \textit{LIBOR Market Models}. \textit{LIBOR} is a short-term\textsuperscript{2} interest rate offered by banks on deposits from other banks in Eurocurrency markets. Since these models were introduced in the market, many financial instruments like \textit{Swaps} and \textit{Caps} which use them have been well developed, and their models such as \textit{Swap Market Models} have important uses as well. It is on these two classes of market models that we focus this work in order to understand them better. We begin with a preliminary chapter in which we review all basic financial tools which we use in the next chapters. In Chapter 2, we build \textit{LIBOR Market Models}. We use thereafter these \textit{LIBOR} interest rates to build \textit{Swap Market Models} in Chapter 3.

Given the complexity of these two types of market models under the \textit{Terminal measure} (which is the last equivalent martingale measure under which they are derived), it is not easy to solve explicitly the stochastic differential equations corresponding to each of these classes of market models. It requires therefore the use of numerical methods to solve them. One widely used numerical method to solve many mathematical problems such as complicated stochastic differential equations is \textit{Monte Carlo Simulations} which we illustrate in Chapter 4 by pricing \textit{caplets} and \textit{caps} within the \textit{LIBOR Market Models}.

\textsuperscript{1}\textit{LIBOR} means London Interbank Offered Rate

\textsuperscript{2}\textit{LIBOR} is usually quoted on 1, 3 or 6-month term
Chapter 1

Elements of Financial Calculus

We begin this work with a brief review of important tools from Probability Theory, Martingales Theory, Itô Calculus and Stochastic Differential Equations that we will use in the next chapters.

1.1 Conditional Expectation

If \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space and \(A, B\) two events such that \(B\) is not a null-event.

We know that the conditional probability of \(A\) given \(B\) is

\[
\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.
\]

Since

\[
\frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{1}{\mathbb{P}(B)} \int_B 1_A d\mathbb{P},
\]

where \(1_A\) is the random variable indicator function on \(A\), we can define \(E(1_A|B) := \frac{1}{\mathbb{P}(B)} \int_B 1_A d\mathbb{P}\) as the conditional expectation of the random variable \(1_A\) given \(B\).

1.1.1 Conditioning on a random variable

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space; \(\mathcal{F}\) is a \(\sigma\)-field of subsets in \(\Omega\) and \(\mathbb{P} : \mathcal{F} \to [0, 1]\) is a probability measure on \(\Omega\).

An integrable random variable on \(\Omega\) is a \(\mathcal{F}\)-measurable function \(X : \Omega \to \mathbb{R}\) such that

\[
\int_{\Omega} |X| d\mathbb{P} < \infty.
\]

Then the integral \(\mathbb{E}_\mathbb{P}(X) := \int_{\Omega} X d\mathbb{P}\) is finite and is called the expectation of the random variable \(X\) under \(\Omega\). We will more often denote \(\mathbb{E}_\mathbb{P}(X)\) simply by \(\mathbb{E}(X)\).

If \(Y\) is an arbitrary random variable with pairwise distinct values \(y_1, y_2, ...\), then the events \(\{Y = y_i\}_{i \geq 1}\) are pairwise disjoint and cover \(\Omega\). We define the sigma-field \(\sigma(Y)\) to be the sub \(\sigma\)-field in \(\mathcal{F}\) generated by those events; \(\sigma(Y)\) is constituted of countable unions of sets of the form \(\{Y = y_n\}\), for \(n\) integer.

Definition 1.1.1

Let \(X\) be an integrable random variable and \(Y\) an arbitrary random variable on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The conditional expectation of \(X\) given \(Y\) is a random variable
\( \mathbb{E}(X|Y) \) such that
(i) \( \mathbb{E}(X|Y) \) is \( \sigma(Y) \)-measurable, and
(ii) For all \( A \) in \( \sigma(Y) \), \( \int_A \mathbb{E}(X|Y) \, d\mathbb{P} = \int_A X \, d\mathbb{P} \).

**Remark 1.1.1**
The conditional expectation of \( X \) given \( Y \) is unique in the sense that if \( X = X' \), then \( \mathbb{E}(X|Y) = \mathbb{E}(X'|Y) \). This uniqueness is a consequence of the lemma below which we will evoke again in the next section.

**Lemma 1.1.1**
Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space and let \( \mathcal{G} \) be a \( \sigma \)-field contained in \( \mathcal{F} \). For every \( \mathcal{G} \)-measurable random variable \( X \), if \( \int_B X \, d\mathbb{P} = 0 \) for every event \( B \) in \( \mathcal{G} \), then \( X = 0 \) a.s.\(^1\)

**Proof.**
For every \( \varepsilon > 0 \), the event \( \{ X \geq \varepsilon \} \) is in \( \mathcal{G} \), and we have
\[
0 \leq \varepsilon \mathbb{P}(\{ X \geq \varepsilon \}) = \int_{\{X \geq \varepsilon\}} \varepsilon \, d\mathbb{P} \leq \int_{\{X \geq \varepsilon\}} X \, d\mathbb{P} = 0;
\]
this implies that \( \mathbb{P}(\{ X \geq \varepsilon \}) = 0 \). Similarly, we have \( \mathbb{P}(\{ X \leq -\varepsilon \}) = 0 \). It follows that \( \mathbb{P}(\{ -\varepsilon < X < \varepsilon \}) = 1 \) for every \( \varepsilon > 0 \).

Now let \( A_n := \{-\frac{1}{n} < X < \frac{1}{n}\} \), where \( n \) is a non-zero integer. It also follows that \( \mathbb{P}(A_n) = 1 \). But since
\[
\{ X = 0 \} = \bigcap_{n=1}^{\infty} A_n,
\]
and the \( A_n \) form a contracting sequence of events, then we have finally
\[
\mathbb{P}(\{ X = 0 \}) = \mathbb{P}\left( \bigcap_{n=1}^{\infty} A_n \right) = \lim_{n \to \infty} \mathbb{P}(A_n) = 1,
\]
Hence \( X = 0 \) a.s. \( \square \^2\)

### 1.1.2 Conditioning on a \( \sigma \)-field

**Definition 1.1.2**
Let \( X \) be an integrable random variable on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and \( \mathcal{G} \) a \( \sigma \)-field contained in \( \mathcal{F} \). The conditional expectation of \( X \) given \( \mathcal{G} \) is the random variable \( \mathbb{E}(X|\mathcal{G}) \) such that
(i) \( \mathbb{E}(X|\mathcal{G}) \) is \( \mathcal{G} \)-measurable, and
(ii) for all \( A \) in \( \mathcal{G} \), \( \int_A \mathbb{E}(X|\mathcal{G}) \, d\mathbb{P} = \int_A X \, d\mathbb{P} \).

---

\(^1\) a.s. means almost surely, that is \( \mathbb{P}(\{ \omega \in \Omega / X(\omega) = 0 \}) = 1 \)

\(^2\) I mean by this box either a proof is ended as usual, either a stated result is true without proof yet, or to sign an important construction we get in this work
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This definition extends Definition 1.1.1 well since if $\mathcal{G} = \sigma(Y)$, then $\mathbb{E}(X|\sigma(Y)) = \mathbb{E}(X|Y)$ a.s. by Lemma 1.1.1.

Some properties of the conditional expectation are:

**Proposition 1.1.1**

Let $X, Y$ be two integrable random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{G}$ a $\sigma$-field contained in $\mathcal{F}$ and $a, b$ two real numbers. Then

(i) $\mathbb{E}(aX + bY|\mathcal{G}) = a\mathbb{E}(X|\mathcal{G}) + b\mathbb{E}(Y|\mathcal{G})$,

(ii) $\mathbb{E}(X|\mathcal{G}) = X$ if $X$ is $\mathcal{G}$-measurable,

(iii) $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$,

(iv) $\mathbb{E}(XY|\mathcal{G}) = X\mathbb{E}(Y|\mathcal{G})$ if $X$ is $\mathcal{G}$-measurable,

(v) $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ if $X$ is independent of $\mathcal{G}$,

(vi) If $X \geq 0$, then $\mathbb{E}(X|\mathcal{G}) \geq 0$.

**Proof.**

The proof uses Lemma 1.1.1 as one can see in [2].

1.2 Martingales in Continuous Time

Martingales are mathematical tools used in pricing market instruments and derivative securities like options. To get an idea from where this concept arose, let us consider the one-step binomial tree model for stock prices below,

![Figure 1.1: One-step binomial tree model](image-url)

At time 0, a stock is worth $S_0$. This price is known by all traders in the market; we can denote this known information at the present time 0 by $\mathcal{F}_0$. But at time 1 in the future, the stock price is $S_1$ and is uncertain. It can go up with the probability $p$ or go down with the probability $1 - p$. $S_1$ is thus a random variable. However, the amount $S_0$ could have been invested in a bank at time 0 attracting interest at a constant rate $r$. This would yield a *risk-less* amount of $(1 + r)S_0$ at time 1.

Using the *arbitrage* construction technique, one can show that the model is *arbitrage-free* (in the sense we will define later) if and only if $S_d < (1 + r)S_0 < S_u$. This is equivalent
to the existence of \( q \in (0, 1) \) such that

\[
(1 + r)S_0 = qS_u + (1 - q)S_d,
\]

\[
S_0 = \frac{qS_u}{1 + r} + (1 - q)\frac{S_d}{1 + r},
\]

\[
S_0 = \mathbb{E}_Q(S'_1), \tag{1.1}
\]

where \( S'_1 = \frac{S_1}{1 + r} \) is called discounted price and \( Q = (q, 1 - q) \) is a probability measure known as risk neutral probability.

The equality (1.1) may be written as

\[
S_0 = \mathbb{E}_Q(S'_1|F_0) \tag{1.2}
\]

Furthermore, if \( X \) is a call option with strike price \( K \) on this stock, that is, a right (but not an obligation) purchased at time 0 to buy the stock at time 1 at a price \( K \), then the payoff\(^3\) of the call will be \( X = (S_1 - K)^+ := \max(S_1 - K, 0) \). Using the Law of one Price (which requires two assets\(^4\) with the same value at a future time for all states of the world, to have the same initial value), one can also get that the price at time 0 of the call option is the expected value under the risk neutral probability \( Q \) of the discounted payoff; that is

\[
X_0 = \mathbb{E}_Q(X'_1) = \mathbb{E}_Q(X'_1|F_0) \tag{1.3}
\]

The two equalities (1.2) and (1.3) suggest a general identity \( X_n = \mathbb{E}(X_{n+1}|F_n) \) that must satisfy a value process \( (X_n)_n \) of assets like stocks or derivative securities\(^5\) such as options in order to define a discrete time martingale.

More generally,

**Definition 1.2.1: Stochastic process**

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. And consider a market in which trading can occur continuously throughout a time line \([0, T]\), where \( T > 0 \).

1) A filtration\(^6\) is the information structure available in the market, that is a family of information up to each time.

Mathematically, a filtration is a family of \( \sigma \)-fields \( \mathcal{F} = (\mathcal{F}_t)_{t \in [0, T]} \) contained in \( \mathcal{F} \) such that

(i) \( \mathcal{F}_0 \) is trivial, that is, contains only all \( \mathbb{P} \)-null sets and their complements,

(ii) \( \mathcal{F}_s \subseteq \mathcal{F}_t \) for all times \( s \leq t \leq T \) (increasing information),

(iii) \( \mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s \) (right continuous),

(iv) \( \mathcal{F}_T = \mathcal{F} \).

Then we say that \((\Omega, \mathcal{F}, \mathbb{P})\) is a filtered probability space.

2) A stochastic process\(^7\) is a sequence of random variables \( X = (X_t)_{t \in [0, T]} \) on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

---

\(^3\)payoff means gain after trading

\(^4\)an asset is any possession that has value in an exchange

\(^5\)market instrument whose price depends on an underlying asset
**Definition 1.2.2: Martingale**

Let \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) be a filtered probability space,

1) An \(\mathbb{F}\)-adapted process is a stochastic process \(X = (X_t)_{t \in [0, T]}\) such that \(X_t\) is \(\mathcal{F}_t\)-measurable for every \(t\) in \([0, T]\).

2) A \(\mathbb{P}\)-martingale or a martingale under \(\mathbb{P}\) is a stochastic process \(X = (X_t)_{t \in [0, T]}\) such that
   
   (i) \(X\) is an \(\mathbb{F}\)-adapted process,
   (ii) \(\mathbb{E}(|X_t|) < \infty\) for all \(t \leq T\),
   (iii) \(\mathbb{E}(X_t | \mathcal{F}_s) = X_s\) for all \(0 \leq s \leq t\) (martingale property)

**Remark 1.2.1**

As \(X_s\) is \(\mathcal{F}_s\)-measurable, then from Proposition 1.1.1 (ii), it follows that the martingale property is equivalent to the identity \(\mathbb{E}(X_t - X_s | \mathcal{F}_s) = 0\) for all times \(0 \leq s \leq t\).

The following result tells that the expectation of a martingale is constant.

**Proposition 1.2.1**

If \(X = (X_t)_{t \in [0, T]}\) is a martingale on a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\), then the expectation \(\mathbb{E}(X_t)\) is constant.

**Proof.**

Let \(t\) in \([0, T]\), then from Proposition 1.1.1 (iii), we have \(\mathbb{E}(X_t) = \mathbb{E}(\mathbb{E}(X_t | \mathcal{F}_0))\) and by the martingale property, \(\mathbb{E}(\mathbb{E}(X_t | \mathcal{F}_0)) = \mathbb{E}(X_0)\). Hence \(\mathbb{E}(X_t) = \mathbb{E}(X_0)\)

**1.3 Brownian Motion and some Properties**

To model the uncertainty of stochastic processes, **Brownian motions** are used to express the randomness on the (relative) change in value of such processes.

**Definition 1.3.1**

A real value stochastic process \(W := (W_t)_{t \geq 0}\) is called a **Standard Brownian motion** or **Wiener process** on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) if

(i) \(W_0 = 0\) a.s.,
(ii) the increment \(W_t - W_s\) follows a normal distribution with mean 0 and variance \(t - s\), for all times \(0 \leq s < t\).
(iii) for all times \(0 < t_1 < ... < t_n\), the random variables \(W_{t_1}, W_{t_2} - W_{t_1}, ..., W_{t_n} - W_{t_{n-1}}\) are independent,
(iv) all simple paths \(t \to W_t(\omega), \omega \in \Omega\), are continuous a.s.

We note in particular that \(\mathbb{E}(W_t) = 0\) and \(\mathbb{E}(W_t^2) = t\) for each time \(t \geq 0\)

It can be shown as in [2] that \(W\) is a Brownian motion if and only if
(i) \( W \) is Gaussian family whose \( W_t \) is normally distributed with mean 0 and variance \( t \),
(ii) \( W_0 = 0 \) a.s. and covariance \( (W_s, W_t) = \min(s, t) \),
(iii) All simple paths \( t \to W_t(\omega) \), \( \omega \in \Omega \), are continuous a.s.

Below is the simulation of one simple path.

\begin{center}
\begin{figure}[h]
\includegraphics[width=\textwidth]{brownian_motion.png}
\caption{A simple path of a Brownian motion}
\end{figure}
\end{center}

**Theorem 1.3.1.** Martingale properties of a Brownian motion

Let \( W = (W_t)_{t \geq 0} \) be a Brownian motion on a probability space \((\Omega, \mathcal{F}, P)\). Consider the filtration \( \mathcal{F} = (\mathcal{F}_t)_{t \geq 0} \), where \( \mathcal{F}_t := \sigma(W_u; u \leq t) \) is the information available up to time \( t \) by observing all \( W_u, u \leq t \). Then the following processes are \( \mathcal{F} \)-martingales

(i) the Brownian motion \( W \) itself,
(ii) \((X_t)_t\) where \( X_t = W_t^2 - t, t \geq 0 \),
(iii) \((Y_t)_t\) where \( Y_t = \exp(\sigma W_t - \frac{1}{2} \sigma^2 t) \) for some constant \( \sigma > 0 \).

**Proof.**

These properties can be derived using the previous equivalent definition of Brownian motion as one may see in [2]

**Remark 1.3.1**

The definition of Brownian motion can be extended to an \( n \)-dimensional Brownian motion as follows.

An \( \mathbb{R}^n \)-valued stochastic process \( W := (W^1, ..., W^n) \) is an \( n \)-dimensional Brownian motion provided

(i) for all \( k = 1, ..., n \), \( W^k \) is a Brownian motion,
(ii) the \( \sigma \)-fields \( \mathcal{G}^k := \sigma(W^k_t; t \geq 0) \) are independent, \( k = 1, ..., n \).

### 1.4 Itô Integral and Itô Calculus

We define a kind of integration with respect to a Brownian motion for a wide class of stochastic processes taken as integrands. As we will see, due to time dependence and
randomness driven by a Brownian motion, this type of integration is different from the usual integration and is a basis of the theory of Stochastic differential equations that we briefly discuss in the next section.

Let \( W = (W_t)_{t \geq 0} \) be a Brownian motion on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with the accompanied filtration \( \mathbb{F} = (\mathcal{F}_t) \), where \( \mathcal{F}_t = \sigma(W_s; 0 \leq s \leq t) \) as previously.

**Definition 1.4.1**

An \( \mathbb{F} \)-adapted process \( X = (X_t)_{t \in [0,T]} \) is called a *step process* on the time interval \([0,T]\) if there exists a partition \( P = \{0 = t_0 < t_1 < \ldots < t_n = T\} \) such that \( X_t \equiv X_k \) is constant for \( t_k \leq t < t_{k+1}; k = 0, \ldots, n - 1 \).

**Definition 1.4.2: Itô Integral of a step process**

Let \( X \) be an adapted step process as above. Then the *Itô Integral* of \( X \) on the interval \([0,T]\) is defined as

\[
\int_0^T X \, dW := \sum_{k=0}^{n-1} X_k (W_{t_{k+1}} - W_{t_k}). \tag{1.4}
\]

Note that this is a random variable.

We can extend this definition to the space \( \mathbb{L}^2(0,T) \) of all real-valued \( \mathbb{F} \)-adapted processes \( X \) such that \( \int_0^T X^2 \, dt < \infty \) a.s in the following way.

If \( X \in \mathbb{L}^2(0,T) \), then as explained in [2], one can approximate the stochastic process \( X \) by a sequence of \( \mathbb{F} \)-adapted step processes \((X^n)_n \) in \( \mathbb{L}^2(0,T) \) such that

\[
\int_0^T |X - X^n|^2 \, dt \to 0 \text{ as } n \to \infty \text{ a.s.}
\]

**Definition 1.4.3**

The *Itô integral* of such a process \( X \) is then defined as

\[
\int_0^T X \, dW := \lim_{n \to \infty} \int_0^T X^n \, dW \tag{1.5}
\]

**Theorem 1.4.1. Some properties of the Itô Integral**

For all \( X, Y \) in \( \mathbb{L}^2(0,T) \) and for all real constants \( a \) and \( b \), then we have

(i) \( \int_0^T (aX + bY) \, dW = a \int_0^T X \, dW + b \int_0^T Y \, dW \)

(ii) \( \mathbb{E} \left( \int_0^T X \, dW \right) = 0 \).

**Proof.**

(i) is obtained from the usual linearity of the integral. Since \( \mathbb{E}(W_t) = 0 \) for all \( t \), we get also (ii) \( \square \)

Let \( \mathbb{L}^1(0,T) \) denote also the space of all \( \mathbb{F} \)-adapted processes \( F \) such that \( \int_0^T |F| \, dt < \infty \) a.s.
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**Definition 1.4.4: Itô process**
A real-valued stochastic process $X$ is called an Itô process if there exist $F$ in $\mathbb{L}^1(0,T)$ and $G$ in $\mathbb{L}^2(0,T)$ such that for all times $0 \leq s \leq r \leq T$, we have

$$X_r = X_s + \int_s^r F dt + \int_s^r G dW$$

we say in this case that $X$ has the Stochastic differential

$$dX = F dt + G dW,$$

for all time $0 \leq t \leq T$.

**Remark 1.4.1: n-dimensional Itô process**
An $n$-dimensional Itô process is a sequence of processes $X = (X^1, ..., X^n)$ such that each process $X^i$ is an Itô process as above, that is, there exist two processes $F^i$ in $\mathbb{L}^1(0,T)$ and $G^i$ in $\mathbb{L}^2(0,T)$ such that, for all times $0 \leq s \leq r \leq T$, we have

$$X^i_r = X^i_s + \int_s^r F^i dt + \int_s^r G^i dW.$$

**Theorem 1.4.2. Itô Formula**
Suppose that $X$ is an Itô process with stochastic differential $dX = F dt + G dW$ for some $F$ in $\mathbb{L}^1(0,T)$ and $G$ in $\mathbb{L}^2(0,T)$. If $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function with continuous partial derivatives $u_t^i$, $u_x^i$, $u_{xx}^i$ such that $u_t^i(t,X_t) \in \mathbb{L}^2(0,T)$. Then the process $Y = (Y_t)_t$ defined by $Y_t := u(t,X_t)$, $t \in [0,T]$, is an Itô process and has the differential

$$dY = \left( u_t^i(t,X_t) + \frac{1}{2} u_{xx}^i(t,X_t) G^2 \right) dt + u_x^i(t,X_t) dX,$$

that is

$$dY = \left( u_t^i(t,X_t) + u_x^i(t,X_t) F + \frac{1}{2} u_{xx}^i(t,X_t) G^2 \right) dt + u_x^i(t,X_t) G dW. \quad (1.6)$$

This equality, which is derived in [2], is known as the Itô Formula. \hfill \Box

More generally,

**Theorem 1.4.3. Generalized Itô Formula**
Let $X = (X^1, ..., X^n)$ be an $n$-dimensional Itô process as above.
If $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function with continuous partial derivatives $u_t^i, u_x^{ij}, u_{xx}^{ij}$ such that $u_t^i(t,X^1, ..., X^n) \in \mathbb{L}^2(0,T)$, $i,j=1, ..., n$. Then the process $Y$ defined by $Y_t := du(t,X_t)$ is also an Itô process, and its stochastic differential is

$$d(u(t,X^1, ..., X^n)) = \left( u_t^i + \frac{1}{2} \sum_{i,j=1}^n u_{xx}^{ij} G^i G^j \right) dt + \sum_{i=1}^n u_x^i dX^i \quad \Box$$
Chapter 1. Elements of Financial Calculus

Example 1.4.1
Let consider the trivial Itô process $X = W$ (that is, the Brownian motion itself); $dX = dW$. So $F \equiv 0$ and $G \equiv 1$. Using the function $u(t, x) := x^m$, $m \geq 2$, the Itô Formula gives

$$d(W^m) = mW^{m-1}dW + \frac{1}{2}m(m-1)W^{m-2}dt$$

In particular when $m = 2$, we have $d(W^2) = 2WdW + dt$. Integrating the two sides in this equality, we get the identity

$$\int_s^r W \, dW = \frac{W_r^2 - W_s^2}{2} - \frac{r - s}{2}$$

for all times $0 \leq s \leq r \leq T$.

This shows well the difference with the usual ordinary integration.

Example 1.4.2
Once again, we take $X = W$, and $u(t, x) := e^{x^2 t}$. Then the Itô Formula gives

$$d\left(e^{W_t - \lambda^2 t^2} \right) = \left(-\frac{\lambda^2}{2}e^{W_t - \lambda^2 t^2} + \frac{\lambda^2}{2}e^{W_t - \lambda^2 t^2} \right) dt + \lambda e^{W_t - \lambda^2 t^2}dW,$$

that is

$$d\left(e^{W_t - \lambda^2 t^2} \right) = \lambda e^{W_t - \lambda^2 t^2}dW.$$

This means that the stochastic process $Y_t := e^{W_t - \lambda^2 t^2}$ is a solution of the system

$$\left\{ \begin{array}{l} dY = \lambda Y \, dW \\ Y_0 = 1 \end{array} \right.$$

which is therefore called a stochastic differential equation.

1.5 Stochastic Differential Equations

Without giving a complete study of Stochastic Differential Equations in this section, we just describe the theory is one-dimension by illustrating some general examples that we will deal with in the next chapters.

Definition 1.5.1
Let $W = (W_t)$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ accompanied by the usual filtration $\mathbb{F} = (\mathcal{F}_t)$, where $\mathcal{F}_t = \sigma(W_s; 0 \leq s \leq t)$.

A real-valued stochastic process $X = (X_t)$ is a solution of the stochastic differential equation

$$\left\{ \begin{array}{l} dX = b(t, X)dt + B(t, X)dW \\ X_0 = x_0 \end{array} \right.$$
for all $0 \leq t \leq T$, if

(i) $X$ is an $\mathbb{F}$-adapted process

(ii) $F := b(t, X) \in L^1(0, T)$

(iii) $G := B(t, X) \in L^2(0, T)$, and we have

(iv) $X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t B(s, X_s)dW_s \text{ a.s. for all } 0 \leq t \leq T$.

Example 1.5.1

Let $f$ and $g$ be deterministic (not random variables) and continuous functions. The stochastic differential equation

\[
\begin{cases}
  dX = fXdt + gXdW \\
  X_0 = 1
\end{cases}
\]

has a solution in the interval $[0, T]$. It is the stochastic process $X_t := e^{\int_0^t (f - \frac{1}{2}g^2)ds + \int_0^t gdw}$. Indeed, consider the Itô process $Y_t := \int_0^t (f - \frac{1}{2}g^2)ds + \int_0^t gdw$. We have $dY = (f - \frac{1}{2}g^2)dt + gdW$, and the Itô Formula for $u(t, x) := e^x$ gives

\[
dX = e^X (f - \frac{1}{2}g^2 + \frac{1}{2}g^2)dt + gdW
\]

that is

\[
dX = X(fdt + gdW)
\]

Example 1.5.2: Black-Scholes model for stock prices

Let $P = (P_t)_t$ be the price process of a stock through a time interval $[0, T]$. We can model the evolution of $P_t$ in time by assuming that the relative change of price $\frac{dP}{P}$ evolves accordingly to the stochastic differential equation

\[
\begin{cases}
  \frac{dP}{P} = \mu dt + \sigma dW \\
  P_0 = p_0
\end{cases}
\]

for some constants $\mu > 0$ and $\sigma$ called the drift and the volatility of the stock price. $\mu$ measures the average change of price in the infinitesimal interval of time $dt$, and $\sigma$ gives the instantaneous deviation in the change of price driven by the Brownian motion $W$.

To solve this stochastic differential equation, we apply Itô Formula with the function $u(t, x) := \log(x)$ and we get

\[
d(\log P) = \left(\frac{1}{P}\mu P - \frac{1}{2} \frac{1}{P^2} \sigma^2 P^2\right)dt + \frac{1}{P}\sigma PdW = (\mu - \frac{\sigma^2}{2})dt + \sigma dW
\]

Consequently, the price process $P$ has the explicit form $P_t = p_0e^{\sigma W_t + (\mu - \frac{\sigma^2}{2})t}$, $t \in [0, T]$.
1.6 Cameron-Martin-Girsanov Theorem

In general, stochastic processes are not martingale under any probability measure. The Cameron-Martin-Girsanov Theorem is used to change the given probability measure into a new measure under which the process becomes a martingale.

Consider a market in which \( n \) assets (like bonds that we will see later) with price processes \( B_1(t), \ldots, B_n(t) \), are traded continuously in a finite interval \([0,T]\). The uncertainty is modelled by a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}, \mathbb{P})\). We also assume that each price process \((B_i(t))_t\) is driven by the same Brownian motion, and follows the Itô process by satisfying the stochastic differential

\[
 dB_i = \mu(t, \omega)dt + \sigma(t, \omega)dW
\]

where \( \mu \in L^1(0,T) \) and \( \sigma \in L^2(0,T) \).

Definitions 1.6.1

(i) A trading strategy is a predictable \( n \)-dimensional stochastic process \( \theta = (\theta_1, \ldots, \theta_n) \) where \( \theta_i(t, \omega) \) denotes the holding in asset with price \( B_i(t) \) at time \( t \), \( i = 1, \ldots, n \). Predictability means, if \( 0 \leq s < t \leq T \), then \( \theta_i(t) \) is \( \mathcal{F}_s \)-measurable.

The value \( V(\theta, t) \) at time \( t \) of a trading strategy is \( V(\theta, t) := \sum_{i=1}^n \theta_i(t)B_i(t) \).

(ii) A self-financing strategy is a trading strategy \( \theta \) whose value process satisfies

\[
 V(\theta, t) = V(\theta, 0) + \sum_{i=1}^n \int_0^t \theta_i(s)dB_i(s),
\]

\( t \in [0,T] \).

We note that as the \( B_i \) follow the Itô process, \( \int_0^t \theta_i(s)dB_i(s) \) are Itô integrals.

(iii) An arbitrage opportunity is a self-financing strategy \( \theta \) such that \( V(\theta, 0) = 0 \) and \( V(\theta, T) \geq 0 \) a.s.

(iv) The market is said arbitrage-free if there is no arbitrage opportunity.

Definition 1.6.2: Equivalent Martingale Measure

(i) A numeraire is an asset \( B \) with strictly positive values \( B(t) \) at any time \( t \) in \([0,T]\)

The role of a numeraire is to discount other asset prices processes \( B_1, \ldots, B_n \) by expressing the relative price processes \( B_i' := B_i/B, i = 1, \ldots, n \). In this work, the numeraires that we consider will mostly be bonds or bank accounts.

(ii) An equivalent martingale measure (associated to a numeraire \( B \)) is a probability measure \( \mathbb{Q} \) on the same filtered probabilisable space \((\Omega, \mathcal{F}, \mathbb{P})\) such that

a) \( \mathbb{Q} \) and \( \mathbb{P} \) have the same null-sets,

b) the discounted price processes \( B_i', i = 1, \ldots, n \), are martingales under \( \mathbb{Q} \).

Theorem 1.6.1. Unique Equivalent Martingale Measure

The market is arbitrage-free if and only if for every choice of numeraire, there exist a
unique equivalent martingale measure.

One may find a proof of this result in [7]. It implies that, in an arbitrage-free market, if we have two numeraires, then we have two different equivalent martingale measures. We can change from one measure to another via the result below also described in [7].

**Theorem 1.6.2. Change of Numeraire**

If $Q$ and $Q^*$ are two equivalent martingale measures associated to the numeraires $B$ and $B^*$ respectively, then for all $A$ in $\mathcal{F}$ we have

$$Q(A) = \int_A \rho(t)dQ^*, $$

where $\rho(t) := \frac{B(T)/B(t)}{B^*(T)/B^*(t)}$ is a random variable called Change of numeraire or Radon-Nikodym derivative and is denoted by $\frac{dQ}{dQ^*}$. □

We end this preliminary chapter by stating one of the most used result in this work, which is also described in [7].

**Theorem 1.6.3. Cameron-Martin-Girsanov Theorem**

Let $g = (g(t))_{t\in[0,T]}$ be a stochastic process such that $\int_0^t g(s)^2ds < \infty$ a.s. Consider the change of numeraire $\frac{dQ^*}{dQ} = \rho(t)$ given by

$$\rho(t) = \exp\left(\int_0^t g(s)dW_s - \frac{1}{2}\int_0^t g(s)^2ds\right),$$

where $Q$ and $Q^*$ are two equivalent martingale measures associated to two numeraires $B$ and $B^*$, and $W$ is a Brownian motion under the measure $Q$. Then, under the measure $Q^*$, the stochastic process $W^*$ defined by

$$W_t^* := W_t - \int_0^t g(s)ds,$$

is a Brownian motion, and satisfies the stochastic differential $dW = dW^* + g(t)dt$ □
Chapter 2

LIBOR Market Models

In this chapter, we model LIBOR interest rates by stochastic differential equations, and we discuss one interest rate derivative\(^1\) known as caps which can be priced under these models.

2.1 Stochastic Interest Rates

If you borrow $1000 today, you will surely not pay back the same amount at a future time. You will certainly pay more than the $1000 borrowed. In financial markets, a formal way to borrow money is via market instruments such as bonds.

A bond is a financial security\(^2\) promising the holder a guaranteed interest payment in the future. There are many kind of bonds. A bond requiring several interest payments in the future is called coupon bond. The one we discuss here is known as zero-coupon bond or simply called a bond. It requires a single interest payment at a future time.

In a bond transaction, there are two counter-parties. The writer (or seller) of the bond borrows from the lender (or buyer of the bond) an amount of money called principal at a certain time. It promises to pay the lender back that principal plus some interest at a fixed date in the future. This future date is usually called the maturity, and we say that the bond matures at that time.

More formally, if at time \(t\) we lend an amount of money denoted by \(B(t, T)\), which has to be paid back at a fixed date \(T\) in the future, how much will we receive back at that time?

In general, the amount received back at time \(T\) is denoted by \(B(T, T)\) and is called face value of the bond. It is determined by means of some interest rate \(r\) on the amount borrowed \(B(t, T)\). But in financial modelling, given an interest rate \(r\), the problem is rather to find the amount \(B(t, T)\) to lend at time \(t\) so that we receive back 1 unit of currency ($1 for example) at time \(T\). \(B(t, T)\) is often called present value of the bond or simply bond price at time \(t\). It depends on the rate \(r\) which can be fixed (also said simple)

\(^1\)An interest rate derivative is a market instrument whose value depends on an underlying interest rate, LIBOR rate in our case
\(^2\)a security is a piece of paper that proves ownership of stocks, bonds, or other investments
or continuously compounded on the interval \([t, T]\) of the loan. It is known (as one can see in [3]) that, when \(r\) is a simple rate, we have

\[
1 = (1 + (T - t)r)B(t, T),
\]

and in the continuous compounding case,

\[
1 = e^{(T-t)r}B(t,T).
\]

This gives

\[
B(t,T) = \frac{1}{1 + (T - t)r},
\]

or

\[
B(t,T) = e^{-(T-t)r}.
\]

**Remark 2.1.1**

We can see how bonds and interest rates are linked. Hence, modelling interest rates is equivalent to modelling bond prices.

### 2.1.1 Simple and Stochastic Interest Rates

Consider a bond with price \(B(t, T)\) as previously in the continuous compounding case for example. If the rate \(r\) is constant through the interval \([t, T]\) of the loan, it is called a simple rate.

But, if \(r := r(t,T)\) depends on the running time \(t\) (subsequently on \(T\)) and randomly on the state \(\omega\) of the world, we call it a floating or stochastic interest rate. This gives the relation

\[
r(t,T) = -\frac{\log B(t,T)}{T - t} \quad (2.1)
\]

In practice, interest rates in bond market are seldom fixed. They are more often stochastic. In this way, the stochastic interest rates we are studying are Forward rates, LIBOR rates and Forward LIBOR rates.

#### 2.1.2 Forward Interest Rates

If a company buys a bond with price \(B(t, T_1)\) at time \(t\) and maturing at time \(T_1\), it should pay back \(e^{(T_1 - t)r(t,T_1)}B(t, T_1)\) at time \(T_1\). Suppose that the company wishes to extend the maturity from \(T_1\) to \(T_2\). At that time \(T_2\), it should pay \(e^{(T_2 - t)r(t,T_2)}B(t, T_2)\). To avoid facing the floating rate \(r(t,T_2)\) which can increase at time \(T_2\), this company can arrange with its counter-party to pay back the amount \(B(T_2, T_2)\) at an agreed rate \(f(t,T_1,T_2)\) quoted on the interval \([T_1, T_2]\) that is, which comes into effect at time \(T_1\) and lasts at time \(T_2\). Then we have

**Definition 2.1.2**
(i) A forward rate in such a contract is the rate \( f \) agreed at time \( t \), coming into effect at time \( T_1 \) and lasting at time \( T_2 \) so that

\[
e^{(T_1 - t) f(t, T_1, T_2)} B(t, T_1) = e^{(T_2 - t) f(t, T_1, T_2)} B(t, T_2),
\]

Solving this, we see that the forward rate in terms of the two bond prices is

\[
f(t, T_1, T_2) = -\frac{\log B(t, T_2) - \log B(t, T_1)}{T_2 - T_1}
\]

(ii) If the increment \( \delta = T_2 - T_1 \) is smaller and smaller, taking \( T_2 = T \), (2.2) would converge to the function

\[
f(t, T) = -\frac{\partial}{\partial T} \log B(t, T)
\]

\( f(t, T) \) is called instantaneous forward rate. It is the rate agreed at time \( t \) and quoted on the infinitesimal interval of time \( [T, T + dt] \).

Given the instantaneous rates \( f(t, T) \), one can recover the present value of the bond as

\[
B(t, T) = \exp \left(-\int_t^T f(t, x)dx\right)
\]

Remark 2.1.1: The Heath-Jarrow-Morton Model

We can introduce randomness on the previous floating rate \( f(t, T) \) by letting it be driven by a Brownian motion \( W \) (on an implicit filtered probability space) such that the process \( f(t, T) \) follows the Itô process

\[
f(t, T) = f(0, T) + \int_0^t \alpha(s, T, \omega)ds + \int_0^t \sigma(s, T, \omega) dW_s,
\]

with stochastic differential

\[
df(t, T) = \alpha(t, T, \omega)dt + \sigma(t, T, \omega)dW_t,
\]

where the processes \( \alpha \) and \( \sigma \) are in \( L^1(0, T) \) and \( L^2(0, T) \) respectively.

(2.5) is called the Heath-Jarrow-Morton Model for instantaneous forward rates. Our goal in this work is not to describe these kind of models completely, but to study similar models more commonly used in the real-world market and which consider a type of interest rates known as LIBOR interest rates.

2.2 LIBOR Market Models

Although the previous Heath-Jarrow Morton models for instantaneous forward rates are mathematically convenient, they are not more often applied in the real-world market since the modelled forward rate \( f(t, T) \) does not exist in practice. Hence, to value financial
securities like bonds and other market instruments like Swaps, (that we study in the next chapter), LIBOR rates have been introduced recently and appeared more practical. We model their dynamics in this section.

Consider a market with a sequence of dates $T_1 < ... < T_n < T_{n+1}$ called tenor structure. In this market, $n+1$ bonds with prices $B(t,T_i)_{1 \leq i \leq n+1}$ at time $t$ are traded, and each of these matures at time $T_i$, $i = 1, ..., n+1$. Let $\delta := T_{i+1} - T_i$ the constant period of the trading interval $[T_i - T_{i+1}]$. $\delta$ is called the tenor.

**Definition 2.2.1: LIBOR interest rate**

LIBOR interest rate is the simple interest rate $L(T_i)$ on the bond price $B(T_i,T_{i+1})$ set at time $t_i$ and remaining fixed up to time $T_{i+1}$. This rate is reset at each time $T_i$ such that the bond price $B(T_i,T_{i+1})$ yields 1 at time $T_{i+1}$. Then we have the identity

$$1 = (1 + \delta L(T_i)) B(T_i,T_{i+1});$$

which gives

$$L(T_i) = \frac{1}{\delta} \left( \frac{1}{B(T_i,T_{i+1})} - 1 \right), i = 1, ..., n. \quad (2.6)$$

**Remark 2.2.1**

If the tenor $\delta$ is small, then using (2.4), the LIBOR interest rate can be expressed in term of the instantaneous forward rates as

$$L(T_i) = \frac{1}{\delta} \exp \left( \int_{T_i}^{T_{i+1}} f(T_i,x) dx \right), i = 1, ..., n. \quad (2.7)$$

**Definition 2.2.2: forward LIBOR rate**

Let consider two different maturities dates $T_1$ and $T_2$ with $T_1 \leq T_2$. $B(t,T_i)$ still denotes the price of a bond at time $t$, maturing at time $T_i$, $i=1,2$.

The Forward LIBOR rate is constructed as follows. At time $t \leq T_1$, we agree to invest the amount $B(t,T_2)$ discounted by $B(t,T_1)$ at time $T_1$ in a bank which pays off 1 at time $T_2$.

The Forward LIBOR rate $L(t,T_1,T_2)$ is the LIBOR rate agreed at time $t$ and quoted on the interval $[T_1,T_2]$ with tenor $\delta_{T_1T_2} := T_2 - T_1$. It comes into effect at time $T_1$ and lasts at time $T_2$. $L(t,T_1,T_2)$ satisfies therefore the equality

$$1 = (1 + \delta_{T_1T_2} L(t,T_1,T_2)) \frac{B(t,T_2)}{B(t,T_1)};$$

which yields the expression of the forward LIBOR rate

$$L(t,T_1,T_2) = \frac{1}{\delta_{T_1T_2}} \left( \frac{B(t,T_1) - B(t,T_2)}{B(t,T_2)} \right) \quad (2.8)$$

**2.2.1 Forward LIBOR Rate Processes**

Consider a market with $n+1$ maturities dates $T_1 < T_2 < ... < T_n < T_{n+1}$. At time $t \leq T_1$, assume we have $n$ forward LIBOR rates $L_i(t) := L(t,T_i,T_{i+1})$ with tenors $\delta_i := \delta_{T_iT_{i+1}}$. 


i = 1, ..., n. We assume also that under a given filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) measuring the information in the market, the associated bonds price processes \(B_i(t) := B(t, T_i)\), \(i = 1, ..., n + 1\) follow Itō processes.

From (2.8), the corresponding Forward LIBOR rates processes are given by

\[
L_i(t) = \frac{1}{\delta_i} \left( \frac{B_i(t)}{B_{i+1}(t)} - 1 \right), \quad i = 1, ..., n
\]  

(2.9)

Our first main goal in this work is to model those Forward LIBOR rates under suitable equivalent martingale measures. For that, we first have the following result.

Theorem 2.2.1

If the market is arbitrage-free, then for every \(i = 1, ..., n\), there exist an equivalent martingale measure denoted by \(\mathbb{Q}^{i+1}\), under which the LIBOR rates process \(L_i(t)\) is a martingale.

**Proof.**

Let consider the bond price \(B_{i+1}(t)\) as a numeraire discounting the Itō process \(B_i(t)\). Then from Theorem 1.6.1, there exist a unique equivalent martingale measure \(\mathbb{Q}^{i+1}\) associated to the numeraire \(B_{i+1}(t)\). By Definition 1.6.2, the discounted Itō process \(\frac{B_i(t)}{B_{i+1}(t)}\) is a martingale under the measure \(\mathbb{Q}^{i+1}\). As \(\delta_i\) is constant, it follows that the Forward LIBOR process \(L_i(t)\) is also a martingale under this measure \(\mathbb{Q}^{i+1}\).

2.2.2 LIBOR Market Models

As each discounted prices process \(\frac{B_i(t)}{B_{i+1}(t)}\) is a martingale under the measure \(\mathbb{Q}^{i+1}\), it turns out that the drift term of the Itō process \(B_i(t)\) disappeared during the change of the measure \(\mathbb{P}\) into its equivalent martingale measure \(\mathbb{Q}^{i+1}\). Therefore as martingales under the measures \(\mathbb{Q}^{i+1}\), the Forward LIBOR rate processes \(L_i(t)\) have no drift term and are assumed to satisfy the Stochastic differentials

\[
dL_i(t) = \sigma_i(t)L_i(t)dW^{i+1}, \quad i = 1, ..., n,
\]  

(2.10)

where \(\sigma_i\) is a deterministic function and \(W^{i+1}\) is the Brownian motion under \(\mathbb{Q}^{i+1}\).

Each of these stochastic differential equations is called the LIBOR Market Model for the Forward LIBOR rate process \(L_i(t)\) under the equivalent martingale measure \(\mathbb{Q}^{i+1}\).

One can solve each of these stochastic differential equations by applying the Itō Formula (Theorem 1.4.2) on the Itō process \(L_i(t)\) using the function \(u(t, x) := \log(x)\). Then it follows that the explicit solution of each stochastic differential equation in (10) is given by

\[
L_i(t) = L_i(0) \exp \left( -\frac{1}{2} \int_0^t \sigma(s)^2 ds + \int_0^t \sigma(s)dW_s^{i+1} \right), \quad i = 1, ..., n.
\]  

(2.11)

**Remark and definition 2.2.1**

When \(i = n\), \(\mathbb{Q}^{n+1}\) is the last equivalent martingale measure; it is called the Terminal measure. As we obtained, under this measure, the last Forward LIBOR rate process \(L_n(t)\) is a martingale. I call it the Terminal Forward LIBOR process.
And for \(i = 1, \ldots, n-1\), the other Forward LIBOR rates processes \(L_i(t)\) are martingales under the correspond measures \(Q^{i+1}\) too. But a fundamental question in this work is what if we express all these other LIBOR rates processes \(L_i(t), i = 1, \ldots, n-1\), under the Terminal measure \(Q^{n+1}\)?

Answering this question will require a technical use of the Cameron-Martin-Girsanov Theorem in the next subsection.

### 2.2.3 LIBOR Market Models under the Terminal Measure

Consider the Change of numeraire \(\rho(t) = \frac{dQ^i}{dQ^{i+1}}\) where the bonds prices \(B_i(t)\) and \(B_{i+1}(t)\) are both taken as numeraires associated to the measures \(Q^i\) and \(Q^{i+1}\) respectively. We know from Theorem 1.6.2 that \(\rho(t)\) has the form

\[
\rho(t) = \frac{B_i(t)/B_i(0)}{B_{i+1}(t)/B_{i+1}(0)}
\]

Using the equality (2.9), we get

\[
\rho(t) = \frac{B_{i+1}(0)}{B_i(0)} (1 + \delta_i L_i(t)).
\]  

(2.12)

Then we have,

**Lemma 2.2.1**

If \(g(t)\) is a stochastic process such that

(i) \(\int_0^t g(s)^2ds < \infty\) a.s.

(ii) \(\rho(t) = \exp(\int_0^t g(s)dW_{s+1} - \frac{1}{2} \int_0^t g(s)^2ds)\),

then the Change of numeraire \(\rho(t)\) satisfies the stochastic differential

\[
d\rho(t) = \rho(t)g(t)dW_{i+1}
\]

**Proof.**

Consider the IT\(\bar{0}\) process \(X := \int_0^t g(s)dW_{s+1} - \frac{1}{2} \int_0^t g(s)^2ds\). Applying the It\(\bar{o}\) Formula with the function \(u(t, x) := e^x\), we get

\[
d\rho(t) = (\frac{1}{2}g(t)^2 + \frac{1}{2}g(t)^2)\rho(t)dt + \rho(t)g(t)dW_{i+1},
\]

hence \(d\rho(t) = \rho(t)g(t)dW_{i+1}\)  

**Theorem 2.2.2.** Passage under a higher order measure

For all \(i = 1, \ldots, n\), the Brownian motions \(W^i\) and \(W^{i+1}\) under their respective equivalent martingale measures \(Q^i\) and \(Q^{i+1}\) satisfy simultaneously the stochastic differential

\[
dW^i = dW^{i+1} - \frac{\delta_i \sigma_i(t)L_i(t)}{1 + \delta_i L_i(t)} dt
\]

(2.13)

**Proof.**

To prove this fundamental result, we need to apply the Cameron-Martin-Girsanov Theorem by finding a stochastic process \(g(t)\) satisfying \(\int_0^t g(s)^2ds < \infty\) a.s. and such that

\[
\rho(t) = \exp(\int_0^t g(s)dW_{s+1} - \frac{1}{2} \int_0^t g(s)^2ds).
\]
If such a process exist, then by the previous lemma, we will have \( d\rho(t) = \rho(t)g(t)dW^{i+1} \). On the other hand, combining (2.12) and (2.10), we get also that

\[
\begin{align*}
d\rho(t) &= \delta_i \frac{B_{i+1}(0)}{B_i(0)} dL_i(t) \\
&= \delta_i \frac{B_{i+1}(0)}{B_i(0)} \sigma_i(t) L_i(t) dW^{i+1} \\
&= \frac{\delta_i \sigma_i(t) L_i(t)}{1 + \delta_i L_i(t)} B_{i+1}(0) (1 + \delta_i L_i(t)) dW^{i+1},
\end{align*}
\]

thus

\[
\frac{\delta_i \sigma_i(t) L_i(t)}{1 + \delta_i L_i(t)} \right) \rho(t) dW^{i+1}
\]

By identification, the process we are seeking is therefore

\[
g(t) = \frac{\delta_i \sigma_i(t) L_i(t)}{1 + \delta_i L_i(t)},
\]

and this process satisfies clearly the hypothesis of the Cameron-Martin-Girsanov Theorem. Hence, applying this one we get the identity

\[
W^i = W^{i+1} - \int_0^t g(s)ds,
\]

that is,

\[
dW^i = dW^{i+1} - \frac{\delta_i \sigma_i(t) L_i(t)}{1 + \delta_i L_i(t)} dt, \ i = 1, ..., n
\]

Hence, under the Terminal measure \( \mathbb{Q}^{n+1} \), using equations (2.10), we have:

When \( i = n - 1 \),

\[
\begin{align*}
dL_{n-1}(t) &= \sigma_{n-1}(t)L_{n-1}(t) dW^n \\
&= \sigma_{n-1}(t) L_{n-1}(t) \left( dW^{n+1} - \frac{\delta_n \sigma_n(t) L_n(t)}{1 + \delta_n L_n(t)} dt \right),
\end{align*}
\]

thus

\[
dL_{n-1}(t) = -\frac{\delta_n \sigma_n(t) L_n(t)}{1 + \delta_n L_n(t)} \sigma_{n-1}(t) L_{n-1}(t) dt + \sigma_{n-1}(t) L_{n-1}(t) dW^{n+1}
\]

is the Market Model for the Forward LIBOR rate process \( L_{n-1}(t) \) under the Terminal measure \( \mathbb{Q}^{n+1} \).

Again for \( i = n - 2 \),

\[
\begin{align*}
dL_{n-2}(t) &= \sigma_{n-2}(t) L_{n-2}(t) \left( dW^n - \frac{\delta_{n-1} \sigma_{n-1}(t) L_{n-1}(t)}{1 + \delta_{n-1} L_{n-1}(t)} dt \right) \\
&= \frac{\delta_{n-1} \sigma_{n-1}(t) L_{n-1}(t)}{1 + \delta_{n-1} L_{n-1}(t)} \sigma_{n-2}(t) L_{n-2}(t) dt + \sigma_{n-2}(t) L_{n-2}(t) \left( dW^{n+1} - \frac{\delta_n \sigma_n(t) L_n(t)}{1 + \delta_n L_n(t)} dt \right),
\end{align*}
\]
we get also that the stochastic differential equation
\[ dL_n(t) = -\left( \frac{\delta_{n-1} \sigma_{n-1}(t)L_{n-1}(t)}{1 + \delta_{n-1} L_{n-1}(t)} + \frac{\delta_n \sigma_n(t)L_n(t)}{1 + \delta_n L_n(t)} \right) \sigma_{n-2}(t)L_{n-2}(t)dt + \sigma_{n-2}(t)L_{n-2}(t)dW^{n+1}, \]
is the Market Model for the Forward LIBOR rate process \( L_{n-2}(t) \) under the same Terminal measure \( Q^{n+1} \).

More generally, still using (2.13) and (2.10), we get by backward induction that, under the Terminal measure \( Q^{n+1} \), the other \( n - 1 \) Forward LIBOR processes \( L_i(t) \) satisfy respectively the stochastic differential equations
\[ dL_i(t) = -\left( \sum_{j=i+1}^{n} \frac{\delta_j \sigma_j(t)L_j(t)}{1 + \delta_j L_j(t)} \right) \sigma_i(t)L_i(t)dt + \sigma_i(t)L_i(t)dW^{n+1}, \quad i = 1, \ldots, n - 1 \quad (2.14) \]
which constitute therefore the LIBOR Market Models for these other Forward LIBOR rates processes under the Terminal measure \( Q^{n+1} \).

**Remark 2.2.2**
To answer the question set in Remark and Definition 2.2.1, we can see from the last stochastic differential equations derived that, under the Terminal measure \( Q^{n+1} \), apart from the Terminal LIBOR process \( L_n(t) \), all the other \( n - 1 \) Forward LIBOR rate processes \( L_i(t), \quad i = 1, \ldots, n - 1 \) contain a non-zero drift term which is
\[ \mu_i(t) := -\sum_{j=i+1}^{n} \frac{\delta_j \sigma_j(t)L_j(t)}{1 + \delta_j L_j(t)} \sigma_i(t)L_i(t), \]
and this one depends on the process \( L_i(t) \) itself. It thus turns out that these \( n - 1 \) first Forward LIBOR rates processes are no longer martingales under the Terminal measure.

Due to the complexity of these drift terms \( \mu_i(t), \quad i = 1, \ldots, n - 1 \), it is not easy to solve explicitly the stochastic differential equations in (2.14). Hence, to price LIBOR rate derivatives such as caps (see next section), it appears better to use numerical methods to solve these equations. Monte Carlo Simulation is then one numerical method that we will study in Chapter 4 to achieve this goal.

### 2.3 Caps and Caplets

**Definition 2.3.1**
A caplet is a provision attached to a LIBOR rate bond and which specifies the maximum interest rate paid over the lifetime of the loan.

If a loan consists of several dates interest payments, and at each date an interest payment is settled with a caplet agreement, then this gives a sequence of caplets which is called a cap.
Consider a company which borrows at forward LIBOR rates and which should make a sequence of interest payments at each date $T_{i+1}$, $i = 1, \ldots, n$. To insure against the possible increasing of the LIBOR rate $L_i(T_i)$ at each time $T_i$, $i = 1, \ldots, n$, this company may purchase a cap contract which requires to pay only the interest at a predetermined fixed rate $K$ at each time $T_{i+1}$ when $L_i(T_i) > K$. Then the company may make an interest rate profit of

$$C_i(T_{i+1}) := (L_i(T_i) - K)^+$$

at each date $T_i$. This profit $C_i(T_{i+1})$ is called the payoff for a single caplet in such a cap contract. The caplet can then be seen as a call option on the LIBOR rate, and the cap as a portfolio of caplets. The value of a cap is therefore

$$V^\text{cap} := \sum_{i=1}^{n-1} C_i(T_{i+1}) = \sum_{i=1}^{n-1} (L_i(T_i) - K)^+$$

### 2.3.1 Cap Value Process under the Terminal Measure

The present value of a caplet at time $t$ is in fact $C_i(t) := (L_i(t) - K)^+$, it is an Itô process since $L_i(t)$ is one. Then, taking the bond price $B_{n+1}(t)$ as a numeraire, the discounted payoff $C_i'(t) := C_i(t)/B_{n+1}(t)$ is a martingale under the Terminal measure $Q^{n+1}$. This discounted payoff is known as numeraire rebased caplet payoff. Its value at time $T_{i+1}$ is thus

$$C_i'(T_{i+1}) := C_i(T_{i+1})/B_{n+1}(T_{i+1}).$$

Then, it follows that the numeraire rebased value of the cap is

$$V^{\text{cap}} := \sum_{i=1}^{n-1} C_i'(T_{i+1}) = \sum_{i=1}^{n-1} \frac{(L_i(T_i) - K)^+}{B_{n+1}(T_{i+1})}$$

By the martingale property (Definition 1.2.2 (iii)), we get also the value of the numeraire rebased payoff at time $T_i$ as

$$C_i'(T_i) = E^{n+1} (C_i'(T_{i+1})|\mathcal{F}_{T_i}),$$

where $E^{n+1}$ denotes the expectation under the Terminal measure $Q^{n+1}$.

We will use this expectation value form in order to calculate the caplet payoffs using the Monte Carlo simulation for LIBOR Market Models in Chapter 4.
Chapter 3

Swap Market Models

In this chapter, we model Swaps using LIBOR interest rates. As in the previous chapter, we will always assume that the given market is arbitrage-free.

3.1 Swaps and Swaptions

Definitions 3.1.1
A swap is an agreement between two parties to exchange a sequence of interest rate payments on some predetermined principal in the future.
A Plain Vanilla swap is a swap in which the two interest rates in question are fixed and floating.

The convention in such swaps is that, in a payer swap denoted by $PS$, the party pays the fixed rate interest and receives the floating rate interest payment. While in a receiver swap denoted $RS$, the party receives the fixed rate interest payment and pays the floating rate interest.

Example 3.1.1
Consider a three-year swap initiated on March 1, 2000, where company $A$ agrees to pay company $B$ at a fixed rate of 5% per annum on the notional principal of $100$ million. Also $B$ agrees to pay $A$ at the six-month LIBOR rate on the same principal. Assume that payments are to be exchanged every six months and the 5% rate is quoted with semi-annual compounding. So in this swap, $A$ is at the payer swap side while $B$ is at the receiver swap side.

The first exchange of interest payments would take place on September 1, 2000. $A$ would pay to $B$ the amount $\frac{1}{2} \times 5\% \times $100 million = $2.5$ million, while $B$ pays $A$ interest on the six-month LIBOR rate prevailing six months prior to September 1, 2000, that is from March 1, 2000. Suppose that LIBOR = 4.2% at that date, then $B$ would pay $\frac{1}{2} \times 4.2\% \times $100 million = $2.1$ million to $A$.

The second exchange of payments would take place on March 1, 2001. $A$ would pay the same fixed amount of $2.5$ million to $B$. And, if LIBOR on September 1, 2000 was
reset at 4.8%, then \( B \) would pay \( A \) $2.4 million to \( A \).

In total there are six exchanges of interest payments on this three-year swap. The fixed rate payments are always $2.5 million since the agreed fixed rate 5% is maintained over the duration of the swap. However, the floating rate payments are recalculated using the six-month LIBOR prevailing at the beginning of each 6-month payment period.

We remark that there is some difference between the fixed rate payment and the floating rate payment at each exchange period. This difference is called value of the swap. For the two first payment exchanges, this value is positive in the receiver swap side and is negative in the payer swap side. In practice, the swap agreement is such that, in the case of this example, company \( B \) may pay only that difference to company \( A \) during the first two exchange periods. This suggests to find in advance the present value of the swap, that is, its value at the running time \( t \).

### 3.1.1 Valuation of Swaps

Consider a sequence of payments dates \( T_1 < T_2 < \ldots < T_n < T_{n+1} \) in a swap where payments are exchanged at each time \( T_i \), \( i = 1, \ldots, n+1 \). These payments can be seen as interests quoted on bonds with prices \( B_i(t) := B_i(t, T_i) \) at the running time \( t \) and maturing at each date \( T_i \). As in Chapter 2, we assume that these bonds \( B_i(t) \) are random and follow the Itô process under the usual filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) where \( \mathbb{F} \) is the accompanied filtration determined by the history of the Brownian motion \( W \) under the measure \( \mathbb{P} \).

On the floating rate side of this swap, we assume that each floating interest payment made at time \( T_{i+1} \) is based on the LIBOR fixing \( \delta_i L_i(T_i) \), where \( \delta_i = T_{i+1} - T_i \) is the usual tenor of that payment trading interval \([T_i, T_{i+1}]\), and \( L_i(t) \) is the forward LIBOR rate studied in the previous chapter. The present value (at the running time \( t \)) of such a single payment made at time \( T_{i+1} \) is therefore

\[
V_{i}^{\text{flo}}(t) := B_{i+1}(t) \mathbb{E}^{i+1}(\delta_i L_i(T_i)),
\]

where \( \mathbb{E}^{i+1} \) denotes the expectation under the unique equivalent martingale measure \( \mathbb{Q}^{i+1} \) (associated to the numeraire \( B_{i+1}(t) \) like in Chapter 2). From Theorem 2.2.1, the LIBOR rate process \( L_i(t) \) is a martingale under the measure \( \mathbb{Q}^{i+1} \). Hence, for \( t \leq T_i \), \( i = 1, \ldots, n+1 \),

\[
\mathbb{E}^{i+1}(\delta_i L_i(T_i)) = \delta_i L_i(t) \text{ by Proposition 1.2.1.}
\]

Then, it follows from (2.9), that the present value of the floating interest payment made at time \( T_{i+1} \) is

\[
V_{i}^{\text{flo}}(t) = B_i(t) - B_{i+1}(t) \tag{3.1}
\]

On the fixed rate side, if each fixed payment made at time \( T_{i+1} \) is quoted at a fixed rate \( K \), then the present value of this single interest payment is

\[
V_{i}^{\text{fix}}(t) = B_{i+1}(t) \delta_i K \tag{3.2}
\]
Chapter 3. Swap Market Models

Definition 3.1.2: Value of the swap
(i) The value of a payer swap at time $t$, starting at time $T_k$, $k = 1, ..., n$, and ending at time $T_{n+1}$ is

$$V_{k,n+1}^{PS} := \sum_{i=k}^{n} V_i^{flo}(t) - \sum_{i=k}^{n} V_i^{fix}(t), k = 1, ..., n,$$

it follows from (3.1) and (3.2) that this payer swap value is

$$V_{k,n+1}^{PS} = (B_k(t) - B_{n+1}(t)) - K \sum_{i=k}^{n} \delta_i B_{i+1}(t), k = 1, ..., n. \quad (3.3)$$

(ii) The value of a receiver swap at time $t$, starting at time $T_k$, $k = 1, ..., n$ and ending at time $T_{n+1}$ is

$$V_{k,n+1}^{RS} := \sum_{i=k}^{n} V_i^{fix}(t) - \sum_{i=k}^{n} V_i^{flo}(t), k = 1, ..., n,$$

Similarly, we get from (3.1) and (3.2) that the receiver swap value is

$$V_{k,n+1}^{RS} = K \sum_{i=k}^{n} \delta_i B_{i+1}(t) - (B_k(t) - B_{n+1}(t)), k = 1, ..., n \quad (3.4)$$

3.1.2 Forward Swap Rates

Definition 3.1.3
The forward swap rate or par swap rate for a swap starting at time $T_k$, $k = 1, ..., n$, and ending at time $T_{n+1}$, is the value of the fixed rate $K$ such that the present value $V_{k,n+1}^{S}(t)$ of the swap is equal to zero. We denote this rate by $y_{k,n+1}(t)$.

Solving $V_{k,n+1}^{PS}(t) = 0$ or $V_{k,n+1}^{RS}(t) = 0$ for $K = y_{k,n+1}$, we find the forward swap rate at the running time $t$ as

$$y_{k,n+1}(t) = \frac{B_k(t) - B_{n+1}(t)}{P_{k+1,n+1}(t)}, k = 1, ..., n, \quad (3.5)$$

where

$$P_{k+1,n+1}(t) := \sum_{i=k+1}^{n+1} \delta_i B_i(t), k = 1, ..., n. \quad (3.6)$$

$P_{k+1,n+1}(t)$ is called Present Value of a Basispoint or PVBP.

Remark 3.1.1
(i) A one-period swap is a swap starting at time $T_i$ and lasting at $T_{i+1}$, $i = 1, ..., n$. Using (3.5), a one-period swap rate $y_{i,i+1}(t)$ is just the LIBOR rate $L_i(t)$. 25
(ii) Combining (3.3), (3.4) and (3.5), and for a given par swap rate $y_{k,n+1}$, one can find the present value of the swap with a different fixed rate $K$ as

$$V_{k,n+1}^{PS}(t) = (y_{k,n+1}(t) - K)P_{k+1,n+1}(t)$$

(3.7)

$$V_{k,n+1}^{RS}(t) = (K - y_{k,n+1}(t))P_{k+1,n+1}(t)$$

(3.8)

(iii) We note that since the PVBP’s $P_{k+1,n+1}(t)$ constitute Itô processes as finite sum of Itô processes $B_i(t)$ under the measure $\mathbb{P}$, and since the forward swap rates $y_{k,n+1}(t)$ are then also Itô processes from (3.5), therefore the value of the swap $V_{k,n+1}^{S}(t)$ is also an Itô process under the probability measure $\mathbb{P}$.

### 3.1.3 Swaption Pricing in a Forward measure

We have seen in the last subsection that the value of a swap varies with time and state. If a company wishes to enter into a swap at a later time, it may purchase an option on that swap. Hence,

**Definition 3.1.4**

A *swaption* is an option to enter into a swap at a given rate in the future.

Consider a company which, at time $t$, gets an option to enter into a receiver swap starting at time $T_k$ and ending at time $T_{n+1}$. The value of this swap at the entrance date $T_k$ that this company may receive over all the interest exchange dates is $V_{k,n+1}^{RS}(T_k)$. If $V_{k,n+1}^{RS}(T_k) < 0$, the company will not exercise its option, that is, it will not enter into the swap. But if this value is positive, the company may receive a positive amount of money and then, will enter into the swap by that time $T_k$. Then,

The value of the (receiver) swaption at time $T_k$ is

$$RS_{k,n+1}(T_k) := (V_{k,n+1}^{RS}(T_k))^+.$$  

(3.9)

The next result enables us to price receiver swaptions under a certain equivalent martingale measure.

**Proposition 3.1.1** There exist a unique equivalent martingale measure $\mathbb{Q}^{k+1,n+1}$ such that

$$RS_{k,n+1}(0) = P_{k+1,n+1}(0)E^{k+1,n+1}(K - y_{k,n+1}(T_k))^+,$$  

(3.10)

where $E^{k+1,n+1}$ denotes the expectation under the measure $\mathbb{Q}^{k+1,n+1}$.

**Proof.**

As a portfolio\(^1\) of bonds, the PVBP $P_{k+1,n+1}$ has strictly positive values. So taking it as a numeraire, it follows from Theorem 1.6.1, that there exist a unique equivalent martingale

---

\(^1\)portfolio means trading strategy in discrete time seen in chapter 1
measure \( Q^{k+1,n+1} \) under which the discounted receiver swap process \( RS_{k,n+1}(t)/P_{k+1,n+1} \) is a martingale. And by Proposition 1.2.1, we have

\[
E^{k+1,n+1} \left( \frac{RS_{k,n+1}(T_k)}{P_{k+1,n+1}} \right) = \frac{RS_{k,n+1}(0)}{P_{k+1,n+1}(0)},
\]

using (3.8) and (3.9), we have in the other hand

\[
RS_{k,n+1}(T_k) \frac{P_{k+1,n+1}(T_k)}{P_{k+1,n+1}} = (K - y_{k,n+1}(T_k))^+. \]

Hence

\[
RS_{k,n+1}(0) = P_{k+1,n+1}(0)E^{k+1,n+1}(K - y_{k,n+1}(T_k))^+. \]

\[\]  

**Corollary 3.1.1**

From this last equality, and under the assumption that the swap rate process \( y_{k,n+1}(t) \) follows a lognormal distribution\(^2\), a straightforward calculation as illustrated in [7,p.94] leads to an explicit form of the receiver swap value as

\[
RS_{k,n+1}(0) = P_{k+1,n+1}(0)E^{k+1,n+1}(K - y_{k,n+1}(T_k))^+ \]

where \( d_1 = (\log(y_{k,n+1}(0)/K) + 1/2\Sigma_{k,n+1}^2)/\Sigma_{k,n+1} \), \( d_2 = d_1 - \Sigma_{k,n+1} \), \( \Sigma_{k,n+1} \) is the standard deviation of \( \log y_{k,n+1}(T_k) \), and \( \mathcal{N} \) denotes the density of the centred-reduced normal distribution\(^3\).

\[\]

**Remark 3.1.2**

Similarly, the value at time 0 of the **payer swaption** can be derived as

\[
PS_{k,n+1}(0) = P_{k+1,n+1}(0)(y_{k,n+1}(0)\mathcal{N}(d_1) - K\mathcal{N}(d_2)) \tag{3.12}
\]

\(PS_{k,n+1}(0)\) and \(RS_{k,n+1}(0)\) are known as **Black** (1976) formula used in the market to price instruments like swaptions.

### 3.2 Swap Market Models

We have seen in Subsection 3.1.2 that the **forward swap rate** \( y_{k,n+1}(t) \) at time \( t \), for a swap starting at time \( T_k \) and ending at time \( T_{n+1} \) is given by

\[
y_{k,n+1}(t) = \frac{B_k(t) - B_{n+1}(t)}{P_{k+1,n+1}(t)}, \quad k = 1, \ldots, n
\]

and we denoted it simply by \( y_k \) through this section.

Under the assumption made on the bond prices processes \( B_i(t), i = 1, \ldots, n+1 \) in Subsection 3.1.1, the interest rates \( y_k, k = 1, \ldots, n \) follow Itô processes under the usual given

\[\]

\(^2\) a distribution whose the logarithm follows a normal distribution

\(^3\) a normal distribution with mean 0 and variance 1
Chapter 3. Swap Market Models

filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\). As in Chapter 2, our main task in this chapter is to model the dynamics of that sequence of swap rates processes under suitable equivalent martingale measures.

**Theorem 3.2.1**

*If the market is arbitrage-free, then for every \(k = 1, \ldots, n\), there exist a unique equivalent martingale measure \(Q^{k+1,n+1}\) under which the forward swap rate process \(y_k(t)\) is a martingale.*

**Proof.**

Let consider again the PVBP \(P_{k+1,n+1}(t)\) as a numeraire. Then, from Theorem 1.6.1, there exist a unique equivalent martingale measure \(Q^{k+1,n+1}\) under which \(\frac{B_k(t) - B_{n+1}(t)}{B_{k+1,n+1}(t)}\) is a martingale. Hence so is \(y_k(t)\)

Then, as a martingale under the measure \(Q^{k+1,n+1}\), the process \(y_k(t), k = 1, \ldots, n\), does not have a drift term. So, we assume under this measure that, the forward swap rate \(y_k(t)\) follow the process

\[
dy_k(t) = \sigma_{k,n+1}(t)y_k(t)dW^{k+1,n+1}, k = 1, \ldots, n, \tag{3.13}
\]

where \(\sigma_{k,n+1}\) is a deterministic function which will be denoted by \(\sigma_k\) and \(W^{k+1,n+1}\) is the Brownian motion under the measure \(Q^{k+1,n+1}\).

Each of these stochastic differential equations is called the *Swap Market Model* for the forward swap rate \(y_k(t)\) under the measure \(Q^{k+1,n+1}\).

### 3.2.1 Swap Market Models under the Terminal Measure

To model market products for which the underlying swaps share the same final payment date \(T_n\), we need to bring like in Chapter 2, all the forward swap rates processes \(y_k, k = 1, \ldots, n\), under the *Terminal measure* \(Q^{n+1,n+1}\). We denote that measure simply by \(Q^{n+1}\), and its corresponding Brownian motion \(W^{n+1,n+1}\) by \(W^{n+1}\).

When \(k = n\), we already know from Theorem 3.2.1 that the *Terminal forward swap rate process* \(y_n(t)\), which is in fact the *Terminal LIBOR rate process* \(L_n(t)\), is a martingale under the *Terminal measure* \(Q^{n+1}\).

Hence, from (3.13), the *Market Model* for \(y_n(t)\) is the stochastic differential equation

\[
dy_n(t) = \sigma_n(t)y_n(t)dW^{n+1} \tag{3.14}
\]

Next, when \(k = n-1\), the technique followed in order to express the forward swap rate process \(y_{n-1}(t)\) under the same *Terminal measure* \(Q^{n+1}\) requires no longer the *Change of numeraire* and the *Cameron-Martin-Girsanov Theorem* like in the LIBOR processes case, but an arduous use of the *Generalized Itô Formula*. For that,
Lemma 3.2.1
The discounted PVBP $P_{n-1,n+1}(t)/B_{n+1}(t)$ verifies the identity
\[
\frac{P_{n-1,n+1}}{B_{n+1}} = \delta_{n-2} + \delta_{n-1}(1 + \delta_{n-2}y_{n-1}) + \delta_n(1 + \delta_{n-1}y_n)(1 + \delta_{n-2}y_{n-1}) \tag{3.15}
\]

Proof.
From (3.5), we have the following recursive relations
\[
B_k - B_{n+1} = y_k P_{k+1,n+1},
\]
which implies
\[
\delta_{k-1}B_k = \delta_{k-1}B_{n+1} + \delta_{k-1}y_k P_{k+1,n+1}.
\]
Since $\delta_{k-1}B_k = P_{k,n+1} - P_{k+1,n+1}$, we get that
\[
P_{k,n+1} - P_{k+1,n+1} = \delta_{k-1}B_{n+1} + \delta_{k-1}y_k P_{k+1,n+1},
\]
and this gives
\[
P_{k,n+1} = \delta_{k-1}B_{n+1} + P_{k+1,n+1}(1 + \delta_{k-1}y_k).
\]
Hence, substituting the similar equalities corresponding to $k = n - 1$ and $k = n$, we have finally
\[
\frac{P_{n-1,n+1}}{B_{n+1}} = \delta_{n-2} + \delta_{n-1}(1 + \delta_{n-2}y_{n-1}) + \delta_n(1 + \delta_{n-1}y_n)(1 + \delta_{n-2}y_{n-1})
\]

Proposition 3.2.1
Under the Terminal measure $Q^{n+1}$, the drift term of the forward swap rate process $y_{n-1}$ is
\[
\mu_{n-1} = -\frac{\delta_n \sigma_n y_n \delta_{n-1} \sigma_{n-1} y_{n-1}}{\delta_{n-1} + \delta_{n}(1 + \delta_{n-1}y_n)} \tag{3.16}
\]

Proof.
Let us consider the interest payment $B_{n+1}$ as a numeraire, then the discounted PVBP $P_{n-1,n+1}/B_{n+1}$ is a martingale under the Terminal measure $Q^{n+1}$. So, the right hand side of (3.15) is a martingale under $Q^{n+1}$ too.

Now, under the measure $Q^{n+1}$, the last two forward swap rates follow the processes of the form
\[
\begin{align*}
  dy_n &= \mu_n dt + \sigma_n y_n dW^{n+1} \\
  dy_{n-1} &= \mu_{n-1} dt + \sigma_{n-1} y_{n-1} dW^{n+1},
\end{align*}
\]
Chapter 3. Swap Market Models

Since we already know that \( y_n(t) \) is a martingale under \( Q^{n+1} \), its drift \( \mu_n \) is zero. And we need to find explicitly the drift \( \mu_{n-1} \) of the process \( y_{n-1}(t) \). Applying the Generalized \( \text{ITô Formula} \) (Theorem 1.4.3) on the two processes \( y_n(t), y_{n-1}(t) \) using the function \[ u(t, x, y) := \delta_{n-2} + \delta_{n-1}(1 + \delta_{n-2}y) + \delta_n(1 + \delta_{n-1}x)(1 + \delta_{n-2}y), \] we get that under the measure \( Q^{n+1} \), the drift part of the stochastic differential \( du(t, y_n, y_{n-1}) \) is

\[
\left( \mu_n u'_x + \mu_{n-1} u'_y + \frac{1}{2} \sigma^2 y^2 u''_{xx} + \frac{1}{2} \sigma^2 y^2 u''_{yy} + \sigma_n y_n \sigma_{n-1} y_{n-1} u''_{xy} \right) dt, \tag{3.17}
\]

where the argument of the partial derivatives of \( u \) is \((t, y_n, y_{n-1})\).

But \( u(t, y_n, y_{n-1}) \) is equal to the right hand side of (3.15) which is a martingale under \( Q^{n+1} \) by Lemma 3.2.1. So, the drift part (3.17) of \( du(t, y_n, y_{n-1}) \) is zero. Hence, solving (3.17) equal to zero for \( \mu_{n-1}(t) \), we get after some calculations on the partial derivatives of \( u \) that

\[ \mu_{n-1} = -\frac{\delta_n \sigma_n y_n \delta_{n-1} \sigma_{n-1} y_{n-1}}{\delta_{n-1} + \delta_n (1 + \delta_{n-1} y_n)}. \]

Hence, under the Terminal measure \( Q^{n+1} \), with this explicit value of \( \mu_{n-1}(t) \), the forward swap rate \( y_{n-1}(t) \) follows the process

\[ dy_{n-1} = \mu_{n-1} dt + \sigma_{n-1} y_{n-1} dW^{n+1}, \tag{3.18} \]

This stochastic differential equation we have derived is therefore the Swap Market Model for the forward swap rate process \( y_{n-1}(t) \) under the Terminal measure \( Q^{n+1} \) \( \square \)

**Remark 3.2.1**

(i) After the Terminal forward swap rate process \( y_n(t) \) which remains a martingale, we have derived also the before last forward swap rate process \( y_{n-1}(t) \) under the same Terminal measure \( Q^{n+1} \). However, the drift term \( \mu_{n-1}(t) \) of that process is not only different to zero, but also dependent on the stochastic processes \( y_n(t) \) and \( y_{n-1}(t) \) itself. Therefore the forward swap rate process \( y_{n-1}(t) \) is no longer a lognormal martingale under the Terminal measure \( Q^{n+1} \).

(ii) Deriving also the first others forward swap rates processes \( y_1, ..., y_{n-2} \), under the Terminal measure \( Q^{n+1} \) requires the similar technique. One can even show by backward induction that the general formula of (3.15) for the discounted \( PVBP \) is

\[
\frac{P_{k,n+1}(t)}{B_{n+1}(t)} = \delta_{k-1} + \sum_{i=k}^n \delta_i \prod_{j=k}^i (1 + \delta_{j-1} y_j(t)), \quad k = 1, ..., n - 1. \tag{3.19}
\]

Taking once more \( B_{n+1}(t) \) as a numeraire, the left hand side of (3.19) is a martingale under the Terminal measure \( Q^{n+1} \). Hence, for \( k = 1, ..., n - 2 \), applying the Generalized \( \text{ITô Formula} \) on the processes \( y_k, ..., y_n \) using the function

\[ u(t, x_1, ..., x_{n-k+1}) = \delta_{k-1} + \sum_{i=k}^n \delta_i \prod_{j=k}^i (1 + \delta_{j-1} x_{n-j+1}), \]

30
one may arrive after arduous calculations, to a complicated expression for the drift $\mu_k(t)$ of the forward swap rate $y_k(t)$. This rate follows the process

$$dy_k(t) = \mu_k(t)dt + \sigma_k(t)dW^{n+1}_n$$

under the same Terminal measure $Q^{n+1}$, but is no longer a martingale under that measure too.
Chapter 4

Monte Carlo Simulations

Given the complexity of the processes in the LIBOR Market Models under the Terminal measure, it is not easy to solve explicitly the stochastic differential equations in (2.14). Hence, if we want to price interest rate derivatives like caps within these models, we can instead use numerical methods. One method which is widely used for market models is Monte Carlo simulations. The goal in this last part of our work is thus to illustrate the use of these numerical methods in pricing caplets by simulating the LIBOR Market Models under the Terminal measure. Let us start with a brief description of the method itself as it is used in the field of Finance.

4.1 Monte Carlo Expectation, Monte Carlo Variance

Also known as the method of statistical trials, the Monte Carlo method makes possible the simulation of every process influenced by randomness. This method is built from the following most important results (see [5, p.4] and [8, p.27]) in the calculus of probabilities.

Theorem 4.1.1. Strong law of large numbers

Let \((X_i)_{i \geq 1}\) be a sequence of independent random variables following the same distribution as an integrable random variable \(X\). then

\[
E(X) = \lim_{n \to \infty} \frac{1}{n} (X_1 + \ldots + X_n) \text{ a.s.,}
\]

\( (4.1) \)

Then, if we want to approximate the expectation \(E(X)\) by the sample mean

\( E(X, n) := \frac{1}{n} (X_1 + \ldots + X_n) \) for \(n\) large enough, this theorem enables to justify the convergence of such an approximation. In this case, the average \(E(X, n)\) is often called the Monte Carlo expectation of the random variable \(X\).

\(^1\)“Monte Carlo” comes from the city of Monte Carlo in the principality of Monaco in France, famous for its gambling house. But, the American mathematicians J. Neyman, S. Ulam and Metropolis (1949) are considered the originators of these numerical methods.
Theorem 4.1.2. Central limit theorem

If \((X_i)_{i \geq 1}\) is a sequence of independent random variables as above. Then for \(n\) large enough, the random variable \(\frac{1}{n}(X_1 + \ldots + X_n)\) follows approximately a normal distribution with expectation \(\mathbb{E}(X)\) and with variance \(\frac{1}{n}\text{Var}(X)\) where \(\text{Var}(X)\) denotes the variance of \(X\).

This result enables to estimate the variance \(\text{Var}(X)\) by the sample variance

\[ \text{Var}(X, n) := \mathbb{E}(X^2, n) - \mathbb{E}(X, n)^2 \]

in the approximation of \(\mathbb{E}(X)\) by \(\mathbb{E}(X, n)\). \(\text{Var}(X, n)\) is known as the Monte Carlo variance of \(X\). There are methods to reduce this variance like the one described in [5, p.9]. This sample variance becomes smaller as \(n\) is larger. However, our goal is not to estimate that variance here, but to use only the convergence provided in the previous theorem in order to implement the Monte Carlo method.

4.2 Monte Carlo Implementation

To use the Monte Carlo method in simulating quantities like caplets and caps within the LIBOR Market Model, one must follow in general two steps.

The first step consists on putting the quantity which we want to calculate in the form of an expected value. This is often simple as in the case of integration. For our case of study, the martingale equality in (2.19) will enables us to achieve this step. However, putting in the form of an expected value can be difficult for some processes. One can instead express such a quantity in the form of an expected value of some functional of that quantity. This alternative is explained well in [5, p.142].

At the second step, we have to calculate a quantity of the form \(\mathbb{E}(X)\), where \(X\) is a random variable. For that, we need to know how to simulate random variables under the distribution of \(X\). Mathematically, we assume that we have a sequence of independent random variables \((X_i)_{i \geq 1}\), all following the distribution of \(X\). \((X_i)_{i \geq 1}\) is then called the sample of the method.

Computationally, we reduce the simulation of \((X_i)_{i \geq 1}\) to that of a sequence of independent random variables following the uniform distribution on a suitable interval. These later are simulated in the computer by suitable random number generators. In our case of study, we use the standard normal random number (denoted by \texttt{randn}) provided in Octave programming in order to simulate a sequence of independent normal distributions.

After these two steps, when the number \(N\) of samplings is large enough (we will consider 1000 000 in our case), it remains only by the Strong law of large numbers, to approximate \(\mathbb{E}(X)\) with

\[ \mathbb{E}(X) \approx \frac{1}{N} (X_1 + \ldots + X_N) \]  \hspace{1cm} (4.2)

Now, as we announced, to price caplets and caps studied in Section 2.3, we are implementing this method in LIBOR Market Models under the Terminal measure \(\mathbb{Q}^{n+1}\).
4.3 Monte Carlo Method for LIBOR Market Models

4.3.1 Implementation of the Method

Under the Terminal measure $Q^{n+1}$, to calculate the different numeraire rebased payoffs for a cap within the LIBOR Market Models in (2.14) using the Monte Carlo simulation, we proceed along the following steps.

*Simulation of the Brownian motion $W^{n+1}$.*

We know from the definition of Brownian motion that the increments $W_{t+\Delta t} - W_t$ are independent random variables which follow the normal distribution with means 0 and variance $\Delta t$. Hence, a path for the Brownian motion $W^{n+1}$ can be constructed as follows. Given the initial value for example $W^{n+1}_0 = 0$ and a time step $T_k$ for $k = 1, 2, ..., n$, as

$$W_{T_k+1}^{n+1} = W_{T_k}^{n+1} + \sqrt{\Delta T} \varepsilon_k,$$

where the $\varepsilon_k$ are independent standard normal random variables. A graph of such a simple path was already illustrated in Figure 1.2. An algorithmic implementation for the approximation (4.3) is detailed in Appendix I.

*Simulation of the LIBOR Market Models.*

After drawing a path of the Brownian motion $W^{n+1}$, we calculate the set of LIBOR rates $L_i(T_k)$ for $i = 0, ..., n$, $k = 0, ..., i$ in the following way. Using the Euler scheme, we discretize the stochastic differential equations in (2.14), by evaluating the LIBOR rates $L_i$ at the cap settlement dates $T_k$. Then we get the scheme

$$L_i(T_{k+1}) = L_i(T_k) - \left( \sum_{j=i+1}^{n} \frac{\delta_j \sigma_j(T_k) L_j(T_k)}{1 + \delta_j L_j(T_k)} \sigma_i(T_k) L_i(T_k) \Delta T + \sigma_i(T_k) L_i(T_k) \left( W_{T_k+1}^{n+1} - W_{T_k}^{n+1} \right) \right).$$

This scheme gives a path for each LIBOR rate $L_i$. We give an algorithmic implementation of that scheme in Appendix II.

Having calculated the set of LIBOR rates $L_i(T_k)$ along the path of the Brownian motion, one can use (2.6) inductively to evaluate at each date $T_k$, the value of the numeraire $B_{n+1}(T_k)$ as

$$B_{n+1}(T_k) = \prod_{j=k}^{n+1} (1 + \delta_j L_j(T_k))^{-1}$$

*Calculating the numeraire rebased caplet payoffs.*

This stage is the effective implementation of the Monte Carlo method. For a given cap, we want to calculate the numeraire rebased caplet payoffs $C'_i(T_{i+1})$ in (2.17). For that, the martingale equality (2.19) enables us to have the $C'_i(T_{i+1})$ in term of an expected value;
Chapter 4. Monte Carlo Simulations

this achieves the first step of the Monte Carlo implementation we described in Section 4.2. The standard normal random variables implying the simulation of the previous LIBOR rates along the paths of the Brownian motion achieve the second step too.

Now, we draw a large number \( N \) of Brownian motion paths. Using (4.4) and (4.5), we get \( N \) sets of LIBOR rates \( L_i(T_k) \) and \( N \) sets of numeraires \( B_{n+1}(T_k) \). For each path \( l, l = 1, \ldots, N \), we evaluate the \( C'_i(T_{i+1}) \) which are equal to \( (L_i(T_i) - K)^+ / B_{n+1}(T_{i+1}) \), and also the corresponding numeraire rebased cap value,

\[
V'_{\text{cap}}^l = \sum_{i=1}^{n-1} C'_i(T_{i+1}).
\]

At the end of the \( N \) samplings, we obtain the estimated numeraire rebased value of the cap as the average

\[
V'_{\text{cap}} \approx \frac{1}{N} \sum_{l=1}^{N} V'_{\text{cap}}^l.
\]

A complete algorithmic implementation to calculate it in Octave programming language is available in Appendix III.

4.3.2 Simulation Results and Application

We consider an example of LIBOR Market Models with semi-annual LIBOR rates by assuming that \( T_k = 0.5k \) for \( k = 0, \ldots, n = 4 \), that is, the time step is \( \Delta T = 0.5 \). The tenor\(^3\) also is taken at \( \delta_k \equiv 0.5 \). We assume a flat initial term-structure of LIBOR rates \( L_i(0) \) at 5\%, and a flat term-structure of volatility \( \sigma_i \) at 20\%. By following the method as implemented in the previous subsection, and for one path of the Brownian motion \( W^{(5)} \) given by (4.3), we get from the scheme (4.4), and (4.5) the sets of LIBOR rates \( L_i(T_k) \) and numeraires \( B_5(T_k) \) in the table below,

<table>
<thead>
<tr>
<th>( T )</th>
<th>( T_0 = 0 )</th>
<th>( T_1 = 0.5 )</th>
<th>( T_2 = 1.0 )</th>
<th>( T_3 = 1.5 )</th>
<th>( T_4 = 2.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta W^{(5)} )</td>
<td>0.30435</td>
<td>-1.0707</td>
<td>0.25633</td>
<td>0.56788</td>
<td></td>
</tr>
<tr>
<td>( L_0(T_k) )</td>
<td>5.000%</td>
<td>5.000%</td>
<td>4.656%</td>
<td>5.000%</td>
<td>4.658%</td>
</tr>
<tr>
<td>( L_1(T_k) )</td>
<td>5.000%</td>
<td>4.658%</td>
<td>5.590%</td>
<td>5.998%</td>
<td></td>
</tr>
<tr>
<td>( L_2(T_k) )</td>
<td>5.000%</td>
<td>4.661%</td>
<td>5.590%</td>
<td>5.998%</td>
<td></td>
</tr>
<tr>
<td>( L_3(T_k) )</td>
<td>5.000%</td>
<td>4.663%</td>
<td>5.595%</td>
<td>6.007%</td>
<td>5.863%</td>
</tr>
<tr>
<td>( L_4(T_k) )</td>
<td>5.000%</td>
<td>4.663%</td>
<td>5.595%</td>
<td>6.007%</td>
<td>5.863%</td>
</tr>
<tr>
<td>( B_5(T_k) )</td>
<td>0.88385</td>
<td>0.91198</td>
<td>0.92063</td>
<td>0.94257</td>
<td>0.97152</td>
</tr>
</tbody>
</table>

Table 4.1: Path of LIBOR rates

We see how two consecutive paths are close together; this because of the flat term-structure of initial LIBOR rates. For the given path of the Brownian motion, if we plot

\(^3\)In real markets, the dates \( T_k \) are not spaced exactly \( \Delta T \) apart in general. The tenors \( \delta_k \) are determined with a specific algorithm for a given market known as the date-roll convention
these two consecutive paths, their graphs will almost be the same. Below are the graphs for the before last and the *Terminal LIBOR rate processes* $L_{n-1}$ and $L_n$ in the case where $n$ is large enough with monthly time step, but from two different paths of the Brownian motion $W^{(5)}$.

![Figure 4.1: A path of $L_{n-1}$](image1.png)  
![Figure 4.2: A path of $L_n$](image2.png)

However, as already mentioned above, if from one path of the Brownian motion $W^{(5)}$, we simulate two consecutive paths for example, the path of the before last and *Terminal LIBOR rates processes* $L_{n-1}$ and $L_n$, we get that these paths evolve in parallel from the common initial condition. They are almost the same as illustrated in the figure below,

![Figure 4.3: Path of the LIBOR rates $L_{n-1}$ and $L_n$](image3.png)
Chapter 4. Monte Carlo Simulations

Next, after this paths simulation comment, we return back to our example of two-year LIBOR rates, that is \( n = 4 \) with semi-annual time step. To illustrate a concrete application of this work, we consider an investor who enters into a cap contract with strike \( K = 5\% \) at time \( t = 0 \) in this particular LIBOR Market. He knows that the cap may protect him against the possible increasing in LIBOR at the dates \( T_1, \ldots, T_4 \), and he wants to find how much interest he may benefit over these four LIBOR resetting dates. For that:

For one path \( l \) of the Brownian motion \( W^{(5)} \) like the one producing the set of LIBOR rates in Table 4.1, using only the LIBOR rates along the diagonal of that table, we calculate the four numeraire rebased caplet payoffs \( C'_i(T_{i+1}) = (L_i(T_i) - K)^+ / B_5(T_{i+1}) \), \( i = 1, \ldots, 4 \), which gives the expected numeraire rebased caplet payoffs \( C'_i(T_i) \) at time \( T_i \) as required from (2.19). Also, we calculate the corresponding numeraire rebased cap value \( V^{'\text{cap}}_l = \sum_{i=1}^{4} C'_i(T_{i+1}) \). We get the table below,

<table>
<thead>
<tr>
<th>( T )</th>
<th>( T_0 = 0 )</th>
<th>( T_1 = 0.5 )</th>
<th>( T_2 = 1.0 )</th>
<th>( T_3 = 1.5 )</th>
<th>( T_4 = 2.0 )</th>
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</thead>
<tbody>
<tr>
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<td>0.30435</td>
<td>-1.0707</td>
<td>0.25633</td>
<td>0.56788</td>
<td></td>
</tr>
<tr>
<td>( L_0(T_k) )</td>
<td>5.000%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( L_1(T_k) )</td>
<td>4.656%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( L_2(T_k) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( L_3(T_k) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( L_4(T_k) )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( B_5(T_k) )</td>
<td>0.88385</td>
<td>0.91198</td>
<td>0.92063</td>
<td>0.94257</td>
<td>0.97152</td>
</tr>
<tr>
<td>( C'_k(T_k) )</td>
<td>0.000000</td>
<td>0.005813</td>
<td>0.006206</td>
<td>0.006206</td>
<td>0.010281</td>
</tr>
<tr>
<td>( V^{'\text{cap}}_l = )</td>
<td>0.0223</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 4.2: Numeraire rebased caplet payoffs

We see from the table that for this single sampling, the investor would pay only at the decreased LIBOR rate \( L_1(T_1) \) at the cap date \( T_2 \), and would benefit a total interest of 2.23\% by paying at the fixed rate \( K = 0.05 \) during the three last dates \( T_3, T_4, \) and \( T_5 \).

This total discounted interest gain of 2.23\% is for one single path \( l \) of the LIBOR rates \( L_i(T_k) \), that is for one state \( \omega \) in the LIBOR Market implicit filtered probability space. But if we do this repetitively for \( N = 1000000 \) samplings (that is for \( N \) states \( \omega \) of the market) by running the Monte Carlo program in Appendix III, we get the estimated numeraire rebased value of the two-year cap as

\[
V^{'\text{cap}} = \frac{1}{1000000} \sum_{l=1}^{1000000} V^{'\text{cap}}_l = 2.5804\% ,
\]

which is therefore the approximate total discounted interest our investor may benefit over the two-year period of this cap contract.
Conclusion

As a beginner in the field of Mathematical and Computational Finance, we have derived under several equivalent martingale measures, the stochastic differential equations modelling LIBOR interest rates, and also swap contracts that use these interest rates. We have seen that under the Terminal measure, these models are more complex for pricing other derivatives like caps and swaptions. This because, apart from the Terminal process for these LIBOR rates and swap rates, their other processes are no longer martingales under this Terminal measure. Therefore, to price caps within LIBOR interest rate models under the Terminal measure, we have resorted to one widely used numerical methods namely Monte Carlo Simulations. Similarly, one could have also simulated swap market models using these numerical methods in order to price related market instruments. The interested reader can see [7, p.104] for the outline of the method to follow.

Solving numerically a differential equation requires in general an algorithmic implementation in a computer language. The one used to perform the task above has been Octave programming language. But, there is one practical language called Excel which is easily used by many people, companies and financial institutions. My future objective in this subject is therefore to continue in studying deeply LIBOR Market Models, their application and their fitting on the real-world market, and possibly their easy implementation in Excel programming language.
Bibliography


Appendices:

Algorithmic Implementations in Octave Programming

Appendix I: Simulation of the Brownian motion

%This program simulates a path of a Brownian motion for about 3-years %and half period with monthly time step.

dT = 1/12;
n = 500;
W(1) = 0;

for k = 1 : n
    T(k) = k*dT;
    if ( i < n )
        W(k+1) = W(k) + sqrt(dT)*randn;
    endif
endfor

plot(T, W,"b")

clear T

clear W
gset nokey

Appendix II: Simulation of LIBOR Market Models

%This program simulates the path of the LIBOR rates for 2-years period with semi-annual time

n = 4;
dT = 0.5;
W(1) = 0;

for i = 1 : n + 1
    d(i) = 0.5;                  %constant tenors
    for k = 1 : n + 1
        s(i,k) = 0.2;            %constant volatilities
    endfor
endfor

A1
\( T(k) = k \cdot dT; \)
if \( k < n+1 \)
\( W(k+1) = W(k) + \sqrt{dT} \cdot \text{randn}; \)
endif
endfor
endfor

\[ L(1:n+1,1) = 0.05 \cdot \textones(n+1,1); \] % initial LIBOR rates
for \( i = n + 1 : -1: 1 \)
for \( k = 1 : i - 1 \)
\[ L(i,k+1) = L(i,k) - \sum((d'.*s(:,k).*L(:,k))/(1 + d'.* L(:,k))(i+1:n+1,1)) \]
\[ *s(i,k)*L(i,k)*dT + s(i,k)*L(i,k)*(W(k+1) - W(k)); \]
endfor
endfor

plot(T, L(n,:), "r;before last libor; ", T, L(n+1,:), "b;Terminal libor; ")

clear T
clear L(n,:)
clear L(n+1,:)

Appendix III: Monte Carlo Implementation

\( \% \)This is the Monte Carlo program calculating the numeraire rebased
\( \% \)value of the cap contract in subsection 4.3.2

\( N = 1\,000\,000; \) % number of samplings
for \( l = 1 : N \)
\( n = 4; \)
\( dT = 0.5; \)
\( W(1) = 0; \)
for \( i = 1 : n + 1 \)
\( d(i) = 0.5; \)
for \( k = 1 : n + 1 \)
\( s(i,k) = 0.2; \)
\( T(k) = k \cdot dT; \)
if ( \( k < n+1 \) )
\( W(k+1) = W(k) + \sqrt{dT} \cdot \text{randn}; \)
endif
endfor
endfor

\[ L(1:n+1,1) = 0.05 \cdot \textones(n+1,1); \] for \( i = n + 1 : -1: 1 \)
for \( k = 1 : i - 1 \)
\[ L(i,k+1) = L(i,k) - \frac{\sum((d'.s(:,k).*L(:,k))./(1 + d'.L(:,k)))(i+1:n+1,1))}{*s(i,k)*L(i,k)*dT + s(i,k)*L(i,k)*(W(k+1) - W(k))}; \]

\text{endfor}
\text{endfor}

\text{for } k = 1 : n+1
\text{prod} = 1;
\text{for } j = k : n+1
\text{prod} = \text{prod}*(1/(1 + d(j)*L(j,k)));
\text{endfor}
B(n+1,k) = \text{prod}; \quad \%\text{calculation of the numeraires}
\text{endfor}

K = 0.05; \quad \%\text{cap strike rate}
V' = 0;
\text{for } i= 1:n
\text{if ( } L(i,i) - K > 0 )
C'(i+1) = (L(i,i) - K)/B(n+1,i+1); \quad \%\text{calculation of the numeraire rebased}
\quad \%\text{caplet payoffs}
\text{endif}
\text{endfor}

V' = \text{sum(C')}; \quad \%\text{a single numeraire rebased cap value}
\text{if (1==1)}
\text{Total} = V';
\text{else}
\text{Total} = \text{Total} + V';
\text{endif}
\text{endfor}

\text{Average} = \text{Total/N} \quad \%\text{final estimate numeraire rebased}
\quad \%\text{value of the cap}